

Pseudo-hyperkähler Geometry and Generalized Kähler Geometry

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Abstract

We discuss the conditions for extra supersymmetry in $N = (2, 2)$ supersymmetric non-linear sigma models described by one left and one right semi-chiral superfield and carrying a pair of non-commuting complex structures. Focus is on linear non-manifest transformations of these fields that have an algebra that closes off-shell. We solve the conditions for invariance of the action and show that a class of these solutions correspond to a bi-hermitian metric of signature $(2, 2)$ and a pseudo-hyperkähler geometry of the target space. This is in contrast to the usual sector of bi-hermitian geometry with commuting complex structures where extra supersymmetries lead to bi-hypercomplex target space geometry.

1 Introduction

The geometry of the target space of supersymmetric non-linear sigma models is dictated by the number of supersymmetries. Investigating the conditions under which it is possible to add extra, non-manifest supersymmetries to a sigma model has been a very direct route to finding new and interesting results in complex geometry. In two dimensions it has led to a complete description¹ of generalized Kähler geometry (GKG) [1]. It may be described in terms of a generalized potential $K(\phi, \bar{\phi}, \chi, \bar{\chi}, \mathbb{X}_{L,R}, \bar{\mathbb{X}}_{L,R})$ which depends on chiral ϕ , twisted chiral χ and left and right semi-chiral $\mathbb{X}_{L,R}$, $N = (2, 2)$ superfields [2].

The special case of generalized hyperkähler geometry is perhaps less well studied, but a description of additional supersymmetries in purely semi-chiral models was treated already in [3]. The models described there contain additional $N = (2, 2)$ superfields that are $N = (4, 4)$ auxiliaries. Below we describe models with $N = (4, 4)$ supersymmetry that closes off-shell without such auxiliary fields.

The target space metric for GKG is positive definite, but the development in our understanding of GKG has a natural extension to the case of an indefinite (generalized) metric [4]. In particular, metrics of neutral signature have recieved increasing attention [5], [6], [7], partly because it has been shown that they arise naturally in the context of string theory [8], [9], [10].² The neutral metric bears some resemblances with Riemannian metrics, which distinguishes it from other metrics of indefinite signatures.

After describing the $N = (4, 4)$ supersymmetry we present the pertinent mathematical background for the neutral hypercomplex structures and then show how a class of such structures arises from potentials in our sigma model setting.

2 Preliminaries

Consider the generalized Kähler potential $K(\mathbb{X}_{L,R}, \bar{\mathbb{X}}_{L,R})$ for the semi-chiral $N = (2, 2)$ superfields $\mathbb{X}_{L,R}$ satisfying

$$\bar{\mathbb{D}}_+ \mathbb{X}_L = 0, \quad \bar{\mathbb{D}}_- \mathbb{X}_R = 0, \quad (2.1)$$

where the supersymmetry algebra is

$$\{\mathbb{D}_+, \bar{\mathbb{D}}_+\} = i\partial_+, \quad \{\mathbb{D}_-, \bar{\mathbb{D}}_-\} = i\partial_-. \quad (2.2)$$

¹Away from singular points.

²It is an interesting fact that target space pseudo-(hyper)kähler geometry arises as a result of world sheet twisted supersymmetry in [9], [10], whereas we discuss standard susy.

The action

$$S = \int K(\mathbb{X}_{L,R}, \bar{\mathbb{X}}_{L,R}) \quad (2.3)$$

has manifest $N = (2, 2)$ supersymmetry.³ One may ask under which conditions the action (2.3) has additional non-manifest supersymmetries to make it $N = (4, 4)$ supersymmetric.⁴ In this note we limit the study to one set of left and right semi-chiral fields and additional supersymmetries linear in those fields. A linear $N = (4, 4)$ supersymmetry transformation on the fields that closes to the susy algebra $[\delta^\pm, \bar{\delta}^\pm]X = i\bar{\epsilon}^\pm \epsilon^\pm \partial_\pm X$ is given by

$$\begin{aligned} \delta \mathbb{X}_L &= i\bar{\epsilon}^+ \bar{\mathbb{D}}_+(\bar{\mathbb{X}}_L + \mathbb{X}_R + \frac{1}{\kappa} \bar{\mathbb{X}}_R) + i\kappa \bar{\epsilon}^- \bar{\mathbb{D}}_- \mathbb{X}_L + i\frac{1}{\kappa} \bar{\epsilon}^- \bar{\mathbb{D}}_- \mathbb{X}_L, \\ \delta \mathbb{X}_R &= i\bar{\epsilon}^- \bar{\mathbb{D}}_-(\bar{\mathbb{X}}_R - (\kappa^2 - 1)\mathbb{X}_L + \frac{\kappa^2 - 1}{\kappa} \bar{\mathbb{X}}_L) - i\kappa \bar{\epsilon}^+ \bar{\mathbb{D}}_+ \mathbb{X}_R - i\frac{1}{\kappa} \bar{\epsilon}^+ \bar{\mathbb{D}}_+ \mathbb{X}_R, \end{aligned} \quad (2.4)$$

where, up to rescalings of fields and transformation parameters, the real κ is the only free parameter. The action is invariant under the transformations provided that K satisfies two complex partial differential equations

$$\begin{aligned} K_{1\bar{1}} - K_{12} - \kappa K_{\bar{1}2} &= 0, \\ (\kappa^2 - 1)K_{2\bar{2}} + K_{12} - \kappa K_{1\bar{2}} &= 0. \end{aligned} \quad (2.5)$$

The indices 1 and 2 denote the partial derivative w.r.t. the left semi-chiral and the right semi-chiral field, respectively. The system (2.5) may be solved by separating variables to give a two-parameter family of solutions

$$K = \tilde{K} e^y, \quad y = \alpha \mathbb{X}_L + \beta \bar{\mathbb{X}}_L + \gamma \mathbb{X}_R + \delta \bar{\mathbb{X}}_R, \quad (2.6)$$

where \tilde{K} is a constant and

$$\gamma = \frac{\alpha\beta}{\alpha + \kappa\beta}, \quad \delta = \frac{\alpha\beta}{\kappa\alpha + \beta}. \quad (2.7)$$

The reason that the solution still depends on two parameters is that the two complex equations in (2.5) have the same imaginary part. For the model to describe bi-hermitian geometry,⁵

$$\det K_{LR} \neq 0, \quad (2.8)$$

³Such actions may describe target space geometries with definite or indefinite signature. We conjecture that many of the properties of GKG, such as the existence of a generalized potential, hold for arbitrary signature bi-hermitian geometries.

⁴The general question under which conditions the semi-chiral fields admit a $N = (4, 4)$ supersymmetry in arbitrary dimension will be addressed in a separate paper [11].

⁵This condition stems from the fact that K is a generating function for certain symplectomorphisms [2]. Alternatively, the conditions is needed to integrate out the $(1, 1)$ auxiliary fields.

where

$$K_{LR} := \begin{pmatrix} K_{12} & K_{1\bar{2}} \\ K_{\bar{1}2} & K_{\bar{1}\bar{2}} \end{pmatrix}. \quad (2.9)$$

Allowing the parameters in y to be complex, the condition (2.8) reads

$$(|\alpha|^2 - |\beta|^2)(|\gamma|^2 - |\delta|^2) \neq 0. \quad (2.10)$$

From the linearity of the conditions (2.5), the solution integrated over the free parameters is again a solution,

$$\int d\alpha d\beta K(\alpha, \beta; \alpha \mathbb{X}_L + \beta \bar{\mathbb{X}}_L + \gamma \mathbb{X}_R + \delta \bar{\mathbb{X}}_R) + c.c. \quad (2.11)$$

Alternatively, solutions are obtained by letting

$$K = F(y) + \bar{F}(\bar{y}), \quad (2.12)$$

where y is defined as above and γ, δ fulfill the above relations and their complex conjugates. Again linearity means that we can integrate such solutions over the free parameters.

3 Neutral hypercomplex structures

Consider a smooth $4n$ -dimensional manifold \mathcal{M} with three real endomorphisms $I, S, T : T\mathcal{M} \leftarrow$. Then (\mathcal{M}, I, S, T) is called a *pseudo-hypercomplex* or *neutral hypercomplex* manifold if the following conditions are fulfilled [12],[13].

- i. (I, S, T) satisfy the algebra of split quaternions,

$$-I^2 = S^2 = T^2 = 1, \quad IS = T = -SI. \quad (3.1)$$

- ii. (I, S, T) are all integrable.⁶ This is equivalent to the vanishing of the Nijenhuis tensor, i.e., if $A \in (I, S, T)$ then

$$N(X, Y) = A^2[X, Y] - A[AX, Y] - A[X, AY] + [AX, AY] = 0 \quad (3.2)$$

for arbitrary vectors $X, Y \in T\mathcal{M}$.

⁶A sufficient condition is that two of the three structures are integrable. The integrability of the third structure is then implicit [14].

Any neutral hypercomplex structure admits a unique torsion-free connection, referred to as the *Obata connection*, such that [14],[15]

$$\nabla I = \nabla S = \nabla T = 0. \quad (3.3)$$

For a neutral hypercomplex structure (\mathcal{M}, I, S, T) with a metric g , we call $(\mathcal{M}, I, S, T, g)$ a *neutral hyperhermitian* structure if and only if

$$g(IX, IY) = -g(SX, SY) = -g(TX, TY) = g(X, Y) \quad (3.4)$$

for all vectors X, Y . A metric satisfying the (skew)hermiticity conditions (3.4) must have signature $(2n, 2n)$. Such a metric is referred to as *neutral*. On oriented 4-manifolds, every neutral hypercomplex structure allows a compatible hyperhermitian metric locally, (implicitly assumed in [12]) and globally after going to a double cover [21].

Given a smooth oriented 4-manifold \mathcal{M} , there are two equivalent sufficient and necessary conditions for \mathcal{M} to admit a neutral hypercomplex structure.

- a) \mathcal{M} admits two complex structures J_+, J_- with the same orientation, such that $\{J_+, J_-\} = 2c$ for c constant with $|c| > 1$ [2].
- b) \mathcal{M} admits a basis of self-dual 2-forms $\Omega_I, \Omega_S, \Omega_T$ and a 1-form Θ (the *Lee-form*) such that [12], [16]

$$d\Omega_i = \Theta \wedge \Omega_i, \quad i = I, S, T. \quad (3.5)$$

The 2-forms are the fundamental forms associated to (I, S, T) . If $\Theta = 0$, Ω_i are closed and define three symplectic forms, and the structure is called *neutral hyperkähler* [17] or *hypersymplectic* [18]. Then there is a metric of signature $(2, 2)$ and the Levi-Civita connection agrees with the Obata connection.

4 Neutral hyperkähler and bi-hermitian geometry.

The target space geometry described by the generalized Kähler potential $K(\mathbb{X}_{L,R}, \bar{\mathbb{X}}_{L,R})$ is generalized Kähler geometry for a positive definite metric g [2][19][20] with generalized hyperkähler as a subclass. When the metric is indefinite of signature (n, n) the corresponding structures has been called generalized pseudo-Kähler and generalized pseudo-hyperkähler [21]. In their bi-hermitian guise [22] these geometries may be given by the data (\mathcal{M}, g, J_\pm) supplemented by the integrability conditions

$$d^c \omega_+ + d^c \omega_- = 0, \quad dd^c \omega_\pm = 0, \quad (4.1)$$

where ω_{\pm} are the canonical two-forms associated with the two complex structures J_{\pm} . The integrability conditions imply the existence of a closed three-form H . Locally it may be written as $H = dB$ for some two-form B . The three-form H also enters the geometry as the torsion in the connections preserving J_{\pm} :

$$\nabla^{\pm} J_{\pm} = 0, \quad \nabla^{\pm} = \nabla^0 \pm \frac{1}{2} g^{-1} H, \quad (4.2)$$

where ∇^0 is the Levi-Civita connection and (4.2) is a consequence of (4.1).

A condition which ensures (neutral) hyperkähler geometry is

$$\{J_+, J_-\} = 2c\mathbb{I}, \quad (4.3)$$

where c is a constant. The resulting geometry is radically different depending on whether $|c| > 1$ or $|c| < 1$. This may be understood from the following relation:

$$([J_+, J_-])^2 = 4(c^2 - 1), \quad (4.4)$$

which makes

$$J \equiv \frac{1}{\sqrt{1 - c^2}} (J_- + cJ_+) \quad (4.5)$$

a complex structure when $c^2 < 1$ and

$$S \equiv \frac{1}{\sqrt{c^2 - 1}} (J_- + cJ_+) \quad (4.6)$$

a local product structure when $c^2 > 1$.

In the first case

$$K \equiv \frac{1}{2\sqrt{1 - c^2}} [J_+, J_-] = \frac{1}{2\sqrt{1 - c^2}} g\Omega^{-1} \quad (4.7)$$

is a third complex structure, and the set $(I \equiv J_+, J, K)$ generate $SU(2)$. In the second case,

$$T \equiv \frac{1}{2\sqrt{c^2 - 1}} [J_+, J_-] = \frac{1}{2\sqrt{c^2 - 1}} g\Omega^{-1} \quad (4.8)$$

is a second product structure and the set $(I \equiv J_+, S, T)$ generate $SL(2, \mathbb{R}) \cong Sp(2)$.

Below, we shall be interested in the second case, i.e. when $|c| > 1$, in which case the geometry corresponding to the set (I, S, T) is called neutral hypercomplex, as reviewed in the previous section.

When a bi-hermitian sigma model is written entirely in terms of semi-chiral fields, as in the case at hand (2.3), the B -field is globally defined and (in a particular gauge) given by

$$B = \Omega \{J_+, J_-\}, \quad (4.9)$$

where

$$\Omega = \frac{1}{2} \begin{pmatrix} 0 & K_{LR} \\ -K_{RL} & 0 \end{pmatrix} \quad (4.10)$$

is a symplectic form. Here K_{LR} is defined in (2.9) and K_{RL} is the transpose of K_{LR} . Since we assume (4.3), the right hand side in (4.9) is equal to $2\Omega c$ and hence $dB = H = 0$. Further, the metric is [2],[23]

$$g = \Omega[J_+, J_-] = 2(\sqrt{c^2 - 1})\Omega T. \quad (4.11)$$

The Lee-form Θ in (3.5) vanishes and

$$\begin{aligned} \Omega_T &= 2(\sqrt{c^2 - 1})\Omega, \\ \Omega_I &= 2(\sqrt{c^2 - 1})\Omega S, \\ \Omega_S &= 2(\sqrt{c^2 - 1})\Omega I. \end{aligned} \quad (4.12)$$

The integrability conditions, which follow from (4.1) are:

$$\nabla^0 I = 0, \quad \nabla^0 S = 0, \quad \nabla^0 T = 0, \quad (4.13)$$

where the connection is now torsion-free. Note that this means that, for this case, the Obata connection (3.3), agrees with the Levi-Civita connection ∇^0 . Also, the $c^2 > 1$ case implies that the metric is indefinite, in fact in the four-dimensional case at hand the signature is $(2, 2)$, often referred to as the metric being neutral, as in Sec. 2 above.

In the four dimensional case we also have

$$J_+ = \begin{pmatrix} J & 0 \\ 2K_{1\bar{1}}K_{RL}^{-1}\sigma_2 & K_{RL}^{-1}JK_{LR} \end{pmatrix}, \quad J_- = \begin{pmatrix} K_{LR}^{-1}JK_{RL} & 2K_{2\bar{2}}K_{LR}^{-1}\sigma_2 \\ 0 & J \end{pmatrix}, \quad (4.14)$$

where

$$J := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}. \quad (4.15)$$

Using this we find that (4.3) is equivalent to

$$(1 + c)|K_{12}|^2 + (1 - c)|K_{1\bar{2}}|^2 = 2K_{1\bar{1}}K_{2\bar{2}}. \quad (4.16)$$

This relation generalizes the Monge-Ampère equation, and, as described in the next section, is satisfied by our solution (2.12).

5 A class of neutral hyperkähler structures

It is an interesting fact that the single solution (2.12) as well as the sum of two solutions of the type appearing in the integrand in (2.11) have

$$c = -\frac{\kappa^2 + 1}{\kappa^2 - 1} \quad (5.1)$$

in order to fulfill the equation (4.16). Since κ is non-zero, $|c| > 1$. The supersymmetry transformations (2.4) break down at $\kappa \rightarrow \pm\infty$, which suppresses the limit $c \rightarrow -1$, i.e., the limit when the complex structures commute.

This means that we have found a two-parameter family of neutral hyperkähler structures with a potential for the geometry given by (2.12). The full set of geometric data, metric, B -field, complex structures and local product structures is expressible in terms of (functions of) the second derivatives of this potential. As for the positive definite case, it should be possible to linearize this structure and find the nonlinear structure as arising from a quotient, but we shall not pursue this issue here.

Using the relations (4.10)-(4.15) we find the following relations for the fundamental two-forms:

$$\begin{aligned} \Omega_T &= (\sqrt{c^2 - 1}) \begin{pmatrix} 0 & K_{LR} \\ -K_{RL} & 0 \end{pmatrix}, \\ \Omega_I &= \begin{pmatrix} 2cK_{1\bar{1}}\sigma_2 & cJ K_{LR} + K_{LRJ} \\ -cK_{RLJ} - J K_{RL} & -2K_{2\bar{2}}\sigma_2 \end{pmatrix}, \\ \Omega_S &= (\sqrt{c^2 - 1}) \begin{pmatrix} 2K_{1\bar{1}}\sigma_2 & J K_{LR} \\ -K_{RLJ} & 0 \end{pmatrix}. \end{aligned} \quad (5.2)$$

As mentioned above, the generalized Kähler potential $K = F(y) + \bar{F}(\bar{y})$ has c constant and independent of the values of the parameters in y . This is seen by a direct calculation of the anticommutator in (4.9) [24].

As an example of the above solution, we can construct a quadratic solution as $K = y^2 + \bar{y}^2$. Modulo terms that are killed by the integration measure ("Kähler gauge transformations"), this takes the form

$$\begin{aligned} K &= \frac{a}{2} \mathbb{X}_L \bar{\mathbb{X}}_L + \frac{b}{2} \mathbb{X}_R \bar{\mathbb{X}}_R + [(a - (\kappa^2 - 1)b + i\kappa d)] \mathbb{X}_L \mathbb{X}_R \\ &\quad + [\frac{1}{\kappa}(a + (\kappa^2 - 1)b + id)] \mathbb{X}_L \bar{\mathbb{X}}_R + c.c. \end{aligned} \quad (5.3)$$

where a, b, d are real and defined by

$$a := \text{Re}(2\alpha\beta), \quad b := \text{Re}(2\gamma\delta), \quad d := \text{Im}(2\alpha\delta + \bar{\beta}\bar{\gamma}). \quad (5.4)$$

We now choose $\kappa = \sqrt{2}$, which implies $c = -3$ in (5.1), and the parameters α, β, γ and δ in a way consistent with the conditions (2.7)⁷:

$$y = \mathbb{X}_L - \nu \bar{\mathbb{X}}_L + \mathbb{X}_R + \nu \bar{\mathbb{X}}_R, \quad (5.5)$$

with $\nu := \sqrt{2} + 1$. For future use we note the split into real and imaginary part $y = \varphi + i\rho$ according to

$$\begin{aligned} 2\varphi &= \sqrt{2} \left(-(\mathbb{X}_L + \bar{\mathbb{X}}_L) + \nu(\mathbb{X}_R + \bar{\mathbb{X}}_R) \right), \\ 2i\rho &= \sqrt{2} \left(\nu(\mathbb{X}_L - \bar{\mathbb{X}}_L) - (\mathbb{X}_R - \bar{\mathbb{X}}_R) \right). \end{aligned} \quad (5.6)$$

The objects needed to find the fundamental two-forms are, according to (5.2)

$$\begin{aligned} K_{1\bar{1}} &= -\nu(F'' + \bar{F}'') := -\nu\Sigma = -K_{2\bar{2}}, \\ K_{LR} &= \nu \begin{pmatrix} \sqrt{2}D - \Sigma & D \\ -D & -(\sqrt{2}D + \Sigma) \end{pmatrix}, \end{aligned} \quad (5.7)$$

where $D := F'' - \bar{F}''$. Note that the fundamental two-forms are all linear in the second derivatives of the potential K . This is true neither for the structures I, S, T nor for the metric. Defining E to be the sum of the metric and B -field, $E = g + B$, and the submatrices according to

$$E = \begin{pmatrix} E_{LL} & E_{LR} \\ E_{RL} & E_{RR} \end{pmatrix}, \quad (5.8)$$

we may use the general formulae in [2] to calculate

$$\begin{aligned} E_{LL} &= \frac{\nu\Sigma}{2|F''|^2} \begin{pmatrix} 2D(\Sigma - \sqrt{2}D) & 3D^2 - \Sigma^2 \\ 3D^2 - \Sigma^2 & -2D(\Sigma + \sqrt{2}D) \end{pmatrix}, \\ E_{LR} &= \frac{\nu}{4|F''|^2} \begin{pmatrix} -(\sqrt{2}D - \Sigma)(5\Sigma^2 - D^2) & -D(3\Sigma^2 + D^2) \\ D(3\Sigma^2 + D^2) & (\sqrt{2}D + \Sigma)(5\Sigma^2 - D^2) \end{pmatrix}, \\ E_{RL} &= \frac{\nu}{4|F''|^2} \begin{pmatrix} (\sqrt{2}D - \Sigma)(\Sigma^2 - 5D^2) & -D(3\Sigma^2 - 7D^2) \\ D(3\Sigma^2 - 7D^2) & -(\sqrt{2}D + \Sigma)(\Sigma^2 - 5D^2) \end{pmatrix}, \\ E_{RR} &= \frac{\nu\Sigma}{2|F''|^2} \begin{pmatrix} 2D(\Sigma - \sqrt{2}D) & \Sigma^2 - 3D^2 \\ \Sigma^2 - 3D^2 & -2D(\Sigma + \sqrt{2}D) \end{pmatrix}. \end{aligned} \quad (5.9)$$

⁷This represents a very particular choice with all the parameters real.

It serves as a gratifying check on our algebra that the B -field calculated from (5.9) is indeed

$$B = -6\Omega. \quad (5.10)$$

(C.f. (4.9)).

To proceed, we need to choose a particular function $F(y)$. The simplest non-trivial choice is $F = y^2$ and corresponds to (5.3) above. It has $D = 0$ and $\Sigma = 4$ and describes flat space with metric

$$g = 4\nu F'' \begin{pmatrix} 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \quad (5.11)$$

where $F'' = 2$. The metric has the expected signature $(--++)$.

A more interesting example results from $F = e^y$ as in (2.6). It has $D = 2ie^\varphi \sin \rho$ and $\Sigma = 2e^\varphi \cos \rho$. The metric will have an exponential pre-factor e^φ , the rest of the metric will depend on the angular variable ρ only.

We may impose further conditions. Requiring the metric to have real entries, e.g., implies $y = \varphi + i\pi n, n \in \mathbb{Z}$, and we recover the same metric as for the quadratic case (5.11) but with $F'' = e^\varphi$.

6 Comments

In this note we have described how neutral hyperkähler structures arise from a certain subclass of bi-hermitian geometries. These bi-hermitian geometries differ from generalized Kähler geometries in the signature of the metric but are also described locally by a single generalized potential. In the literature we have found one similar statement: In [25], Prop.3 states that every hyperhermitian metric locally arises from a pair of complex valued functions satisfying a certain non-linear differential equation involving both first and second derivatives of the potentials. We have not investigated the exact relation to our (real) generalized potential.

On compact complex surfaces, the neutral hyperkähler structures have been classified [17]. They are 4-tori or Kodaira surfaces. An analysis of which functions $F(y)$ lead to compact complex surfaces is therefore of great interest.

It is further well-known that there exists an infinite family of $4d$ neutral hyperkähler structures [21]. In our examples the arbitrariness in choice of function F in (2.12) again leads to an infinite family of such structures.

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Abstract

We discuss the conditions for additional supersymmetry and twisted supersymmetry in $N = (2, 2)$ supersymmetric non-linear sigma models described by one left and one right semi-chiral superfield and carrying a pair of non-commuting complex structures. Focus is on linear non-manifest transformations of these fields that have an algebra that closes off-shell. We find that additional linear supersymmetry has no interesting solution, whereas additional linear *twisted* supersymmetry has solutions with interesting geometrical properties. We solve the conditions for invariance of the action and show that these solutions correspond to a bi-hermitian metric of signature $(2, 2)$ and a pseudo-hyperkähler geometry of the target space.

1 Introduction

The geometry of the target space of supersymmetric non-linear sigma models is dictated by the number of supersymmetries. Investigating the conditions under which it is possible to add extra, non-manifest supersymmetries to a sigma model has been a very direct route to finding new and interesting results in complex geometry. In two dimensions it has led to a complete description¹ of generalized Kähler geometry (GKG) [1]. It may be described in terms of a generalized potential $K(\phi, \bar{\phi}, \chi, \bar{\chi}, \mathbb{X}_{L,R}, \bar{\mathbb{X}}_{L,R})$ which depends on chiral ϕ , twisted chiral χ and left and right semi-chiral $\mathbb{X}_{L,R}$, $N = (2, 2)$ superfields [2].

The special case of generalized hyperkähler geometry is perhaps less well studied, but a description of additional supersymmetries in purely semi-chiral models was treated already in [3]. The models described there contain additional $N = (2, 2)$ superfields that are $N = (4, 4)$ auxiliaries. Below we describe models with $N = (4, 4)$ (twisted) supersymmetry that closes off-shell without such auxiliary fields.

The target space metric for GKG is positive definite, but the development in our understanding of GKG has a natural extension to the case of an indefinite (generalized) metric [4]. In particular, metrics of neutral signature have recieved increasing attention [5], [6], [7], partly because it has been shown that they arise naturally in the context of string theory [8], [9], [10]. The neutral metrics bear some resemblance to Riemannian metrics, which distinguishes it from other metrics of indefinite signatures.

In this paper we restrict to four dimensional target space and find that additional supersymmetry cannot be imposed. However, we find a class of interesting solutions with additional *twisted* supersymmetry. After describing the $N = (4, 4)$ twisted supersymmetry we present the pertinent mathematical background for the neutral hypercomplex structures and then show how a class of such structures arises from potentials in our sigma model setting.

2 Ansatz

Consider the generalized Kähler potential $K(\mathbb{X}_{L,R}, \bar{\mathbb{X}}_{L,R})$ for the semi-chiral $N = (2, 2)$ superfields $\mathbb{X}_{L,R}$ satisfying

$$\bar{\mathbb{D}}_+ \mathbb{X}_L = 0, \quad \bar{\mathbb{D}}_- \mathbb{X}_R = 0, \quad (2.1)$$

¹Away from singular points.

where the supersymmetry algebra is

$$\{\mathbb{D}_+, \bar{\mathbb{D}}_+\} = i\partial_+, \quad \{\mathbb{D}_-, \bar{\mathbb{D}}_-\} = i\partial_-. \quad (2.2)$$

The action

$$S = \int K(\mathbb{X}_{L,R}, \bar{\mathbb{X}}_{L,R}) \quad (2.3)$$

has manifest $N = (2, 2)$ supersymmetry.² One may ask under which conditions the action (2.3) has additional non-manifest symmetries to make it $N = (4, 4)$ supersymmetric or twisted supersymmetric.

Supersymmetry can be generalized to twisted supersymmetry [9], [10], where some of the generators close to a pseudo-supersymmetry,

$$\{Q^I, Q^J\} = 2\eta^{IJ}P, \quad \eta^{IJ} = \begin{pmatrix} 1_p & 0 \\ 0 & -1_q \end{pmatrix}, \quad (2.4)$$

where P is the translation operators, $p + q = r$ and $I = 1, \dots, r$.

In this note we limit the study to four-dimensional target space, where we have only one set of left and right semi-chiral fields and also restrict the additional transformations to be linear in those fields. The general question under which conditions the semi-chiral fields admit a $N = (4, 4)$ (twisted) supersymmetry in arbitrary dimension is addressed in a separate paper [11].

A general linear transformation that preserves the chirality of the fields reads

$$\begin{aligned} \delta\mathbb{X}_L &= i\bar{\epsilon}^+\bar{\mathbb{D}}_+(\varepsilon\bar{\mathbb{X}}_L + b\mathbb{X}_R + c\bar{\mathbb{X}}_R) + i\kappa\bar{\epsilon}^-\bar{\mathbb{D}}_-\mathbb{X}_L - i\lambda\bar{\epsilon}^-\bar{\mathbb{D}}_-\mathbb{X}_L \\ \delta\bar{\mathbb{X}}_L &= -i\epsilon^+\mathbb{D}_+(\bar{\varepsilon}\mathbb{X}_L + \bar{b}\bar{\mathbb{X}}_R + \bar{c}\mathbb{X}_R) - i\bar{\kappa}\epsilon^-\mathbb{D}_-\bar{\mathbb{X}}_L + i\bar{\lambda}\epsilon^-\mathbb{D}_-\bar{\mathbb{X}}_L \\ \delta\mathbb{X}_R &= i\bar{\epsilon}^-\bar{\mathbb{D}}_-(\tilde{\varepsilon}\bar{\mathbb{X}}_R + \tilde{b}\mathbb{X}_L + \tilde{c}\bar{\mathbb{X}}_L) + i\tilde{\kappa}\bar{\epsilon}^+\bar{\mathbb{D}}_+\mathbb{X}_R - i\tilde{\lambda}\bar{\epsilon}^+\bar{\mathbb{D}}_+\mathbb{X}_R \\ \delta\bar{\mathbb{X}}_R &= -i\epsilon^-\mathbb{D}_-(\tilde{\bar{\varepsilon}}\mathbb{X}_R + \tilde{\bar{b}}\bar{\mathbb{X}}_L + \tilde{\bar{c}}\mathbb{X}_L) - i\tilde{\bar{\kappa}}\epsilon^+\mathbb{D}_+\bar{\mathbb{X}}_R + i\tilde{\bar{\lambda}}\epsilon^+\mathbb{D}_+\bar{\mathbb{X}}_R. \end{aligned} \quad (2.5)$$

We now ask whether this ansatz can close to a supersymmetry or a twisted supersymmetry, and whether the transformations keep the action (2.3) invariant.

2.1 Supersymmetry

The transformations (2.5) close to a supersymmetry algebra

$$[\delta(\epsilon_1), \delta(\epsilon_2)]\mathbb{X} = i\bar{\epsilon}_{[2}^{\pm}\epsilon_{1]}^{\pm}\partial_{\pm\pm}\mathbb{X} \quad (2.6)$$

²Such actions may describe target space geometries with definite or indefinite signature. We conjecture that many of the properties of GKG, such as the existence of a generalized potential, hold for arbitrary signature bi-hermitian geometries.

only if the left- and right sectors decouple. This means that the left semi-chiral field possesses only a left-going supersymmetry, and the right semi-chiral field a right-going. The action (2.3) is invariant under the supersymmetry

$$\begin{aligned}\delta\mathbb{X}_L &= i\kappa\bar{\epsilon}^-\bar{\mathbb{D}}_-\mathbb{X}_L + \frac{i}{\kappa}\epsilon^-\mathbb{D}_-\mathbb{X}_L \\ \delta\mathbb{X}_R &= i\tilde{\kappa}\bar{\epsilon}^+\bar{\mathbb{D}}_+\mathbb{X}_R + \frac{i}{\tilde{\kappa}}\epsilon^+\mathbb{D}_+\mathbb{X}_R\end{aligned}\tag{2.7}$$

if and only if the potential $K(\mathbb{X}_L, \bar{\mathbb{X}}_L, \mathbb{X}_R, \bar{\mathbb{X}}_R)$ is linear in all the fields. But since the sigma model (2.3) will vanish for any linear potential, no solution for additional supersymmetry exists.

2.2 Twisted supersymmetry

On the other hand, the transformations can close to a *pseudo*-supersymmetry

$$[\delta(\epsilon_1), \delta(\epsilon_2)]\mathbb{X} = -i\bar{\epsilon}_{[2}^{\pm}\epsilon_1^{\pm}\partial_{\pm\pm}\mathbb{X}\tag{2.8}$$

for a larger class of solutions, possessing interesting geometric properties. Closing the pseudo-supersymmetry for the transformations (2.5), most of the parameters can be solved for. Further, rescaling the fields and transformation parameters in a manner compatible with R-symmetry, the complex κ will be the only free parameter. The action is invariant under the twisted supersymmetry transformations³

$$\begin{aligned}\delta\mathbb{X}_L &= i\bar{\epsilon}^+\bar{\mathbb{D}}_+(\bar{\mathbb{X}}_L + \mathbb{X}_R + \frac{1}{\kappa}\bar{\mathbb{X}}_R) + i\kappa\bar{\epsilon}^-\bar{\mathbb{D}}_-\mathbb{X}_L - \frac{i}{\kappa}\epsilon^-\mathbb{D}_-\mathbb{X}_L, \\ \delta\mathbb{X}_R &= i\bar{\epsilon}^-\bar{\mathbb{D}}_-(\bar{\mathbb{X}}_R - (\kappa\bar{\kappa} - 1)\mathbb{X}_L + \frac{\kappa\bar{\kappa}-1}{\bar{\kappa}}\bar{\mathbb{X}}_L) - i\bar{\kappa}\bar{\epsilon}^+\bar{\mathbb{D}}_+\mathbb{X}_R + \frac{i}{\bar{\kappa}}\epsilon^+\mathbb{D}_+\mathbb{X}_R,\end{aligned}\tag{2.9}$$

provided that K satisfies two complex partial differential equations

$$\begin{aligned}K_{1\bar{1}} - K_{12} - \bar{\kappa}K_{\bar{1}2} &= 0, \\ (\kappa\bar{\kappa} - 1)K_{2\bar{2}} + K_{12} - \kappa K_{1\bar{2}} &= 0.\end{aligned}\tag{2.10}$$

The indices 1 and 2 denote the partial derivative w.r.t. the left semi-chiral and the right semi-chiral field, respectively. The system (2.10) may be solved by separating variables to give a two-parameter family of solutions

$$K = F(y) + \bar{F}(\bar{y}), \quad y = \alpha\mathbb{X}_L + \beta\bar{\mathbb{X}}_L + \gamma\mathbb{X}_R + \delta\bar{\mathbb{X}}_R,\tag{2.11}$$

³The lack of symmetry between the transformations of the left and right fields may seem puzzling but is just an artifact of our choice of rescalings. Symmetric choices are possible.

where

$$\gamma = \frac{\alpha\beta}{\alpha + \bar{\kappa}\beta}, \quad \delta = \frac{\alpha\beta}{\kappa\alpha + \beta}. \quad (2.12)$$

The reason that the solution still depends on two parameters is that the two complex equations in (2.10) have the same imaginary part. For the model to describe bi-hermitian geometry,⁴

$$\det K_{LR} \neq 0, \quad (2.13)$$

where

$$K_{LR} := \begin{pmatrix} K_{12} & K_{1\bar{2}} \\ K_{\bar{1}2} & K_{\bar{1}\bar{2}} \end{pmatrix}. \quad (2.14)$$

For the solution (2.11), this condition is equivalent to

$$(|\alpha|^2 - |\beta|^2)(|\gamma|^2 - |\delta|^2) \neq 0. \quad (2.15)$$

From the linearity of the conditions (2.10), the solution integrated over the free parameters is again a solution,

$$\int d\alpha d\beta K(\alpha, \beta; \alpha\mathbb{X}_L + \beta\bar{\mathbb{X}}_L + \gamma\mathbb{X}_R + \delta\bar{\mathbb{X}}_R). \quad (2.16)$$

3 Neutral hypercomplex structures

Consider a smooth $4n$ -dimensional manifold \mathcal{M} with three real endomorphisms $I, S, T : T\mathcal{M} \leftarrow$. Then (\mathcal{M}, I, S, T) is called a *pseudo-hypercomplex* or *neutral hypercomplex* manifold if the following conditions are fulfilled [12],[13].

- i. (I, S, T) satisfy the algebra of split quaternions,

$$-I^2 = S^2 = T^2 = 1, \quad IS = T = -SI. \quad (3.1)$$

- ii. (I, S, T) are all integrable.⁵ This is equivalent to the vanishing of the Nijenhuis tensor, i.e., if $A \in (I, S, T)$ then

$$N(X, Y) = A^2[X, Y] - A[AX, Y] - A[X, AY] + [AX, AY] = 0 \quad (3.2)$$

for arbitrary vectors $X, Y \in T\mathcal{M}$.

⁴This condition stems from the fact that K is a generating function for certain symplectomorphisms [2]. Alternatively, the conditions is needed to integrate out the $(1, 1)$ auxiliary fields.

⁵A sufficient condition is that two of the three structures are integrable. The integrability of the third structure is then implicit [14].

Any neutral hypercomplex structure admits a unique torsion-free connection, referred to as the *Obata connection*, such that [14],[15]

$$\nabla I = \nabla S = \nabla T = 0. \quad (3.3)$$

For a neutral hypercomplex structure (\mathcal{M}, I, S, T) with a metric g , we call $(\mathcal{M}, I, S, T, g)$ a *neutral hyperhermitian* structure if and only if

$$g(IX, IY) = -g(SX, SY) = -g(TX, TY) = g(X, Y) \quad (3.4)$$

for all vectors X, Y . A metric satisfying the (skew)hermiticity conditions (3.4) must have signature $(2n, 2n)$. Such a metric is referred to as *neutral*. On oriented 4-manifolds, every neutral hypercomplex structure allows a compatible hyperhermitian metric locally, (implicitly assumed in [12]) and globally after going to a double cover [21].

Given a smooth oriented 4-manifold \mathcal{M} , there are two equivalent sufficient and necessary conditions for \mathcal{M} to admit a neutral hypercomplex structure.

- a) \mathcal{M} admits two complex structures J_+, J_- with the same orientation, such that $\{J_+, J_-\} = 2c$ for c constant with $|c| > 1$ [2].
- b) \mathcal{M} admits a basis of self-dual 2-forms $\Omega_I, \Omega_S, \Omega_T$ and a 1-form Θ (the *Lee-form*) such that [12], [16]

$$d\Omega_i = \Theta \wedge \Omega_i, \quad i = I, S, T. \quad (3.5)$$

The 2-forms are the fundamental forms associated to (I, S, T) . If $\Theta = 0$, Ω_i are closed and define three symplectic forms, and the structure is called *neutral hyperkähler* [17] or *hypersymplectic* [18]. Then there is a metric of signature $(2, 2)$ and the Levi-Civita connection agrees with the Obata connection.

4 Neutral hyperkähler and bi-hermitian geometry.

The target space geometry described by the generalized Kähler potential $K(\mathbb{X}_{L,R}, \bar{\mathbb{X}}_{L,R})$ is generalized Kähler geometry for a positive definite metric g [2][19][20] with generalized hyperkähler as a subclass. When the metric is indefinite of signature (n, n) the corresponding structures has been called generalized pseudo-Kähler and generalized pseudo-hyperkähler [21]. In their bi-hermitian guise [22] these geometries may be given by the data (\mathcal{M}, g, J_\pm) supplemented by the integrability conditions

$$d^c\omega_+ + d^c\omega_- = 0, \quad dd^c\omega_\pm = 0, \quad (4.1)$$

where ω_{\pm} are the canonical two-forms associated with the two complex structures J_{\pm} . The integrability conditions imply the existence of a closed three-form H . Locally it may be written as $H = dB$ for some two-form B . The three-form H also enters the geometry as the torsion in the connections preserving J_{\pm} :

$$\nabla^{\pm} J_{\pm} = 0, \quad \nabla^{\pm} = \nabla^0 \pm \frac{1}{2} g^{-1} H, \quad (4.2)$$

where ∇^0 is the Levi-Civita connection and (4.2) is a consequence of (4.1).

A condition which ensures (neutral) hyperkähler geometry is

$$\{J_+, J_-\} = 2c\mathbb{I}, \quad (4.3)$$

where c is a constant. The resulting geometry is radically different depending on whether $|c| > 1$ or $|c| < 1$. This may be understood from the following relation:

$$([J_+, J_-])^2 = 4(c^2 - 1), \quad (4.4)$$

which makes

$$J \equiv \frac{1}{\sqrt{1 - c^2}} (J_- + cJ_+) \quad (4.5)$$

a complex structure when $c^2 < 1$ and

$$S \equiv \frac{1}{\sqrt{c^2 - 1}} (J_- + cJ_+) \quad (4.6)$$

a local product structure when $c^2 > 1$.

In the first case

$$K \equiv \frac{1}{2\sqrt{1 - c^2}} [J_+, J_-] = \frac{1}{2\sqrt{1 - c^2}} g\Omega^{-1} \quad (4.7)$$

is a third complex structure, and the set $(I \equiv J_+, J, K)$ generate $SU(2)$. In the second case,

$$T \equiv \frac{1}{2\sqrt{c^2 - 1}} [J_+, J_-] = \frac{1}{2\sqrt{c^2 - 1}} g\Omega^{-1} \quad (4.8)$$

is a second product structure and the set $(I \equiv J_+, S, T)$ generate $SL(2, \mathbb{R}) \cong Sp(2)$.

Below, we shall be interested in the second case, i.e. when $|c| > 1$, in which case the geometry corresponding to the set (I, S, T) is called neutral hypercomplex, as reviewed in the previous section.

When a bi-hermitian sigma model is written entirely in terms of semi-chiral fields, as in the case at hand (2.3), the B -field is globally defined and (in a particular gauge) given by

$$B = \Omega \{J_+, J_-\}, \quad (4.9)$$

where

$$\Omega = \frac{1}{2} \begin{pmatrix} 0 & K_{LR} \\ -K_{RL} & 0 \end{pmatrix} \quad (4.10)$$

is a symplectic form. Here K_{LR} is defined in (2.14) and K_{RL} is the transpose of K_{LR} . Since we assume (4.3), the right hand side in (4.9) is equal to $2\Omega c$ and hence $dB = H = 0$. Further, the metric is [2],[23]

$$g = \Omega[J_+, J_-] = 2(\sqrt{c^2 - 1})\Omega T. \quad (4.11)$$

The Lee-form Θ in (3.5) vanishes and

$$\begin{aligned} \Omega_T &= 2(\sqrt{c^2 - 1})\Omega, \\ \Omega_I &= 2(\sqrt{c^2 - 1})\Omega S, \\ \Omega_S &= 2(\sqrt{c^2 - 1})\Omega I. \end{aligned} \quad (4.12)$$

The integrability conditions, which follow from (4.1) are:

$$\nabla^0 I = 0, \quad \nabla^0 S = 0, \quad \nabla^0 T = 0, \quad (4.13)$$

where the connection is now torsion-free. Note that this means that, for this case, the Obata connection (3.3), agrees with the Levi-Civita connection ∇^0 . Also, the $c^2 > 1$ case implies that the metric is indefinite, in fact in the four-dimensional case at hand the signature is $(2, 2)$, often referred to as the metric being neutral, as in Sec. 2 above.

In the four dimensional case we also have

$$J_+ = \begin{pmatrix} J & 0 \\ K_{RL}^{-1}C_{LL} & K_{RL}^{-1}J K_{LR} \end{pmatrix}, \quad J_- = \begin{pmatrix} K_{LR}^{-1}J K_{RL} & K_{LR}^{-1}C_{RR} \\ 0 & J \end{pmatrix}, \quad (4.14)$$

where

$$J := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad C_{LL} = 2iK_{1\bar{1}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad C_{RR} = 2iK_{2\bar{2}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (4.15)$$

Using this we find that (4.3) is equivalent to

$$(1 + c)|K_{12}|^2 + (1 - c)|K_{1\bar{2}}|^2 = 2K_{1\bar{1}}K_{2\bar{2}}. \quad (4.16)$$

This relation generalizes the Monge-Ampère equation, and, as described in the next section, is satisfied by our solution (2.11).

5 A class of neutral hyperkähler structures

Combining the equations (2.10) and (4.16) yields

$$c = -\frac{\kappa\bar{\kappa} + 1}{\kappa\bar{\kappa} - 1}. \quad (5.1)$$

Since κ is non-zero, $|c| > 1$. The twisted supersymmetry transformations (2.9) break down at $|\kappa| \rightarrow \pm\infty$, which suppresses the limit $c \rightarrow -1$, i.e., the limit when the complex structures commute.

This means that we have found a two-parameter family of neutral hyperkähler structures with a potential for the geometry given by (2.11). The full set of geometric data, metric, B -field, complex structures and local product structures is expressible in terms of (functions of) the second derivatives of this potential. As for the positive definite case, it should be possible to linearize this structure and find the nonlinear structure as arising from a quotient, but we shall not pursue this issue here.

Using the relations (4.10)-(4.15) we find the following relations for the fundamental two-forms:

$$\begin{aligned} \Omega_T &= (\sqrt{c^2 - 1}) \begin{pmatrix} 0 & K_{LR} \\ -K_{RL} & 0 \end{pmatrix}, \\ \Omega_I &= \begin{pmatrix} cC_{LL} & cJ K_{LR} + K_{LRJ} \\ -cK_{RLJ} - J K_{RL} & -C_{RR} \end{pmatrix}, \\ \Omega_S &= (\sqrt{c^2 - 1}) \begin{pmatrix} C_{LL} & J K_{LR} \\ -K_{RLJ} & 0 \end{pmatrix}. \end{aligned} \quad (5.2)$$

As mentioned above, the generalized Kähler potential has c constant and independent of the values of the parameters in y . This is seen by a direct calculation of the anticommutator in (4.9) [24]. A constant c implies from (4.9) that the torsion vanishes,

$$dB = 2c \cdot d\Omega = 0. \quad (5.3)$$

5.1 Examples

We now choose $\kappa = \sqrt{2}$, which implies $c = -3$ in (5.1), and the parameters α, β, γ and δ in a way consistent with the conditions (2.12)⁶:

$$y = \mathbb{X}_L - \nu \bar{\mathbb{X}}_L + \mathbb{X}_R + \nu \bar{\mathbb{X}}_R, \quad (5.4)$$

⁶This represents a very particular choice with all the parameters real.

with $\nu := \sqrt{2} + 1$. For future use we note the split into real and imaginary part $y = \varphi + i\rho$ according to

$$\begin{aligned} 2\varphi &= \sqrt{2} \left(-(\mathbb{X}_L + \bar{\mathbb{X}}_L) + \nu(\mathbb{X}_R + \bar{\mathbb{X}}_R) \right), \\ 2i\rho &= \sqrt{2} \left(\nu(\mathbb{X}_L - \bar{\mathbb{X}}_L) - (\mathbb{X}_R - \bar{\mathbb{X}}_R) \right). \end{aligned} \quad (5.5)$$

The objects needed to find the fundamental two-forms are, according to (5.2)

$$K_{1\bar{1}} = -\nu(F'' + \bar{F}'') := -\nu\Sigma = -K_{2\bar{2}}, \quad (5.6)$$

$$K_{LR} = \nu \begin{pmatrix} \sqrt{2}D - \Sigma & D \\ -D & -(\sqrt{2}D + \Sigma) \end{pmatrix}, \quad (5.7)$$

where $D := F'' - \bar{F}''$. Note that the fundamental two-forms are all linear in the second derivatives of the potential K . This is true neither for the structures I, S, T nor for the metric. Defining E to be the sum of the metric and B -field, $E = g + B$, and the submatrices according to

$$E = \begin{pmatrix} E_{LL} & E_{LR} \\ E_{RL} & E_{RR} \end{pmatrix}, \quad (5.8)$$

we may use the general formulae in [2] to calculate

$$\begin{aligned} E_{LL} &= \frac{\nu\Sigma}{2|F''|^2} \begin{pmatrix} 2D(\Sigma - \sqrt{2}D) & 3D^2 - \Sigma^2 \\ 3D^2 - \Sigma^2 & -2D(\Sigma + \sqrt{2}D) \end{pmatrix}, \\ E_{LR} &= \frac{\nu}{4|F''|^2} \begin{pmatrix} -(\sqrt{2}D - \Sigma)(5\Sigma^2 - D^2) & -D(3\Sigma^2 + D^2) \\ D(3\Sigma^2 + D^2) & (\sqrt{2}D + \Sigma)(5\Sigma^2 - D^2) \end{pmatrix}, \\ E_{RL} &= \frac{\nu}{4|F''|^2} \begin{pmatrix} (\sqrt{2}D - \Sigma)(\Sigma^2 - 5D^2) & -D(3\Sigma^2 - 7D^2) \\ D(3\Sigma^2 - 7D^2) & -(\sqrt{2}D + \Sigma)(\Sigma^2 - 5D^2) \end{pmatrix}, \\ E_{RR} &= \frac{\nu\Sigma}{2|F''|^2} \begin{pmatrix} 2D(\Sigma - \sqrt{2}D) & \Sigma^2 - 3D^2 \\ \Sigma^2 - 3D^2 & -2D(\Sigma + \sqrt{2}D) \end{pmatrix}. \end{aligned} \quad (5.9)$$

It serves as a gratifying check on our algebra that the B -field calculated from (5.9) is indeed

$$B = -6\Omega. \quad (5.10)$$

(Cf. (4.9)).

To proceed, we need to choose a particular function $F(y)$. The simplest non-trivial choice is the quadratic solution $F = y^2$ and corresponds to

$$K = \text{Re}(\alpha\beta)\mathbb{X}_L\bar{\mathbb{X}}_L + \text{Re}(\gamma\beta)\mathbb{X}_R\bar{\mathbb{X}}_R + [\alpha\gamma + \bar{\beta}\bar{\delta}]\mathbb{X}_L\mathbb{X}_R + [\alpha\delta + \bar{\beta}\bar{\gamma}]\mathbb{X}_L\bar{\mathbb{X}}_R + c.c., \quad (5.11)$$

modulo terms that are killed by the integration measure ("Kähler gauge transformations"). It has $D = 0$ and $\Sigma = 4$ and describes flat space with metric

$$g = 8\nu \begin{pmatrix} 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}. \quad (5.12)$$

The metric has the expected signature $(- - ++)$.

A more interesting example results from $F = e^y$. It has $D = 2ie^\varphi \sin \rho$ and $\Sigma = 2e^\varphi \cos \rho$. The metric will have an exponential pre-factor e^φ , the rest of the metric will depend on the angular variable ρ only.

We may impose further conditions. Requiring the metric to have real entries, e.g., implies $y = \varphi + i\pi n, n \in \mathbb{Z}$, and we recover the same metric as for the quadratic case (5.12) but with $F'' = e^\varphi$.

6 Comments

In this note we have investigated additional linear symmetries for a $N = (2, 2)$ sigma model with semi-chiral fields restricted to four-dimensional target space. We have found that no solution for additional supersymmetry exists, but that a class of interesting solutions for $N = (4, 4)$ twisted supersymmetry can be found.

We have described how neutral hyperkähler structures arise from a certain subclass of bi-hermitian geometries. In particular, they are described locally by a single generalized potential. A similar statement can be found in [25], where Prop. 3 states that every hyperhermitian metric locally arises from a pair of complex valued functions satisfying a certain non-linear differential equation involving both first and second derivatives of the potentials. We have not investigated the exact relation to our (real) generalized potential.

On compact complex surfaces, the neutral hyperkähler structures have been classified [17]. They are 4-tori or Kodaira surfaces. An analysis of which functions $F(y)$ lead to compact complex surfaces is therefore of great interest.

It is further well-known that there exists an infinite family of $4d$ neutral hyperkähler structures [21]. In our examples the arbitrariness in choice of function F in (2.11) again leads to an infinite family of such structures.

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